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Jordan Algebra and Field Theory

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It is shown that the set of observable functionals associated with a constrained field theory satisfying two given assumptions is a Jordan algebra under the symmetric Dirac bracket composition law.

1. INTRODUCTION

In a recent paper Truini (1986) has shown the usefulness of the concept of Jordan pairs in the study of supergravity theories. Such a concept generalizes that of a Jordan algebra and links it naturally to a Lie algebra.

The aim of the present note is to show that a Jordan algebra (P. Jordan *et al.*, 1934; Albert, 1934; Jacobson, 1949, 1968), realized through the symmetric Dirac brackets (Dirac, 1950, 1958, 1964; Franke and Kálnay, 1970), is an algebraic structure always present in the theory of classical fields.

As is known (Pedroza and Vianna, 1980) for discrete systems we cannot always define a Jordan product using the plus Dirac brackets. Hence it is not evident that the set of observable functionals of a field theory described by the symmetric formulation of classical mechanics satisfy the axioms of a Jordan algebra. Nevertheless, we show that this is true for a wide class of field theories. This class of fields includes half-integer spin (Fermi) systems described by Weyl, Dirac, and Rarita-Schwinger fields.

2. CONSTRAINED FIELD THEORIES

Let us suppose that our field system is described by a function of time t,

$$\Psi = \Psi(\mathbf{x}) = \Psi(\mathbf{x};\, t) = (\Psi_1, \Psi_2, \ldots, \Psi_n)$$

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where $\mathbf{x}=(x_1, x_2, x_3)$ denotes the spatial coordinates of a point in real physical space and Ψ_i $(i=1,2,\ldots,n)$ are the field components. The field velocity n-tuple is then the associated form

$$\dot{\Psi}=\partial\Psi/\partial t=(\dot{\Psi}_1, \dot{\Psi}_2, \ldots, \dot{\Psi}_n)$$

Consider that the Lagrangian $\mathscr{L}=\mathscr{L}[\Psi, \dot{\Psi}]$ is a real scalar functional of $\Psi=\Psi(\mathbf{x})$ and $\dot{\Psi}=\dot{\Psi}(\mathbf{x})$. We have the field momentum density n-tuple

$$\begin{aligned}\Pi=\Pi(\mathbf{x}) &\equiv \delta\mathscr{L}/\delta\dot{\Psi}(\mathbf{x})\\ &=(\delta\mathscr{L}/\delta\dot{\Psi}_1(\mathbf{x}),\ \delta\mathscr{L}/\delta\delta\dot{\Psi}_2(\mathbf{x}), \ldots, \delta\mathscr{L}/\delta\dot{\Psi}_n(\mathbf{x}))\\ &\equiv (\Pi_1(\mathbf{x}), \Pi_2(\mathbf{x}), \ldots, \Pi_n(\mathbf{x}))\end{aligned}$$

where $\delta/\delta\dot{\Psi}_i(\mathbf{x})$ denotes the functional derivative with respect to $\dot{\Psi}_i(\mathbf{x})$; the canonical field variables are $\Pi(\mathbf{x})$ and $\Psi(\mathbf{x})$.

As was shown by Droz-Vincent (1966) for the unconstrained classical system and by Franke and Kálnay (1970) for systems involving constraints, besides the skew-symmetric algebraic structure there exists another symmetric structure. This new classical structure is characterised by the existence of a bracket $\{\ ,\ \}_+$ called the plus (or symmetric) Poisson bracket and a bracket $\{\ ,\ \}_+^*$ called the plus (or symmetric) Dirac bracket, for unconstrained and constrained classical systems, respectively. Hence, for classical fields described by the symmetric classical structure we can define the plus Poisson bracket

$$\{F, G\}_+=\int [\delta F/\delta\Pi_i(\mathbf{x})\cdot\delta G/\delta\Psi_i(\mathbf{x})+\delta F/\delta\Psi_i(\mathbf{x})\cdot\delta G/\delta\Pi_i(\mathbf{x})]\,dx \qquad (1)$$

where $dx=dx_1\,dx_2\,dx_3$ and F and G are observable functionals, and the summation convention with respect to repeated indices is used here and always in the following.

By using a symplectic notation, we can write (1) as

$$\{F, G\}_+=\int S^{A}{}_{i}{}^{B}{}_{j}\,\delta F/\delta\omega^{A}{}_{i}(\mathbf{x})\cdot\delta G/\delta\omega^{B}{}_{j}(\mathbf{x})\,dx$$

$$A, B=1,2; \qquad i,j=1,2,\ldots,n \qquad (2)$$

where $S^{1}{}_{i}{}^{2}{}_{j}=S^{2}{}_{i}{}^{1}{}_{j}=\delta_{ij}$, $S^{1}{}_{i}{}^{1}{}_{j}=S^{2}{}_{i}{}^{2}{}_{j}=0$, $\omega^{1}{}_{i}=\Pi_i$, $\omega^{2}{}_{i}=\Psi_i$.

In the following, for the sake of notational simplicity, we denote $\omega^{A}{}_{i}(\mathbf{x})$ by $\omega^{I}(\mathbf{x})$. Then the relation (2) can be written

$$\{F, G\}_+=\int S^{IJ}\,\delta F/\delta\omega^{I}(\mathbf{x})\cdot\delta G/\delta\omega^{J}\,dx$$

If $\mathscr{L}$ is a degenerate functional Lagrangian (Hermann, 1970; Hanson *et al.* 1976), one has

1. A set of constraints (Dirac, 1950)

$$K_a(\Pi, \Psi) \approx 0, \qquad a = 1, 2, \ldots, m$$

2. A subset of symmetric second-class constraints (Franke and Kálnay, 1970)

$$\Theta_\alpha(\Pi, \Psi) \approx 0, \qquad \alpha = 1, 2, \ldots, p, \qquad m \geq p$$

where p is no longer necessarily even, in opposition to the skew-symmetric formulation. Now it is possible to introduce the plus Dirac bracket (Franke and Kálnay, 1970)

$$\{F, G\}_+^* = \{F, G\} - \int \{F, \Theta_\alpha(\mathbf{x})\}_+ \mathbf{D}_{\alpha\beta}(\mathbf{x}-\mathbf{x}')\{\Theta_\beta(\mathbf{x}'), G\}_+ \, dx\, dx' \tag{3}$$

where $\mathbf{D}_{\alpha\beta}(\mathbf{x}-\mathbf{x}')$ is the inverse matrix of $\|\{\Theta_\alpha(\mathbf{x}), \Theta_\beta(\mathbf{x}')\}_+\|$.

Then it follows readily that we can write (3) as

$$\{F, G\}_+^* = \int M^{IJ}(\mathbf{x}-\mathbf{x}')\, \delta F/\delta\omega^I(\mathbf{x}) \cdot \delta G/\delta\omega^J(\mathbf{x}')\, dx\, dx'$$

with $M^{IJ}(\mathbf{x}-\mathbf{x}')$ depending upon the symmetric second-class constraints, and $M^{IJ}(\mathbf{x}-\mathbf{x}') = M^{IJ}(\mathbf{x}'-\mathbf{x})$ as a consequence of the symmetric property of the brackets $\{\ ,\ \}_+^*$.

We want to know if the set of all dynamical variables $(F(\Pi, \Psi), G(\pi, \Psi), \ldots, H(\Pi, \Psi))$ is a Jordan algebra with the Jordan product defined by the combination law (3). We suppose that:

Assumption 1. The observables $F, G, \ldots, H$ of the continuous dynamical system described by the symmetric formulation of classical field theory are functionals at most quadratic in canonical field variables, i.e.,

$$\begin{aligned} \delta^3 F/\delta\omega^I(\mathbf{x})\, \delta\omega^J(\mathbf{x}')\, \delta\omega^K(\mathbf{x}'') = 0 \\ \delta^3 G/\delta\omega^I(\mathbf{x})\, \delta\omega^J(\mathbf{x}')\, \delta\omega^K(\mathbf{x}'') = 0 \end{aligned} \tag{4}$$

Assumption 2. The symmetric second-class constraints are homogeneous functionals of first degree of the canonical field variables.

As a consequence, from Assumption 2 we have that the elements $M^{IJ}(\mathbf{x}-\mathbf{x}')$ do not depend on any canonical field variables, i.e., $\delta M^{IJ}(\mathbf{x}-\mathbf{x}')/\delta\omega^L(\mathbf{x}) = 0$.

From Assumption 1, we have that if two observables F and G, for example, satisfy the relations (4), $\{F, G\}_+^*$ also satisfies this assumption and the set $(F(\Pi, \Psi),\ G(\Pi, \Psi), \ldots, H(\Pi, \Psi))$ is closed under the product defined by combination law (3). To analyze the identity

$$\{F^2, \{G, F\}_+^*\}_+^* = \{\{F^2, G\}_+^*, F\}_+^* \tag{5}$$

with $F^2 = \{F, F\}_+^*$, which must be satisfied if $(F, G, \ldots, H)$ is a Jordan algebra, we obtain from the left-hand side of (5)

$$\begin{aligned}
\mathscr{I}_1 &= \{F^2, \{G, F\}_+^*\}_+^* \\
&= 2 \int M^{IJ}(\mathbf{x}-\mathbf{x}')M^{KL}(\mathbf{y}-\mathbf{y}')M^{PQ}(\mathbf{z}-\mathbf{z}')\, \delta^2 F/\delta\omega^I(\mathbf{x})\, \delta\omega^K(\mathbf{y}) \\
&\quad \times [\delta F/\delta\omega^J(\mathbf{x}') \cdot \delta^2 G/\delta\omega^L(\mathbf{y}')\, \delta\omega^P(z)\, \delta F/\delta\omega^Q(\mathbf{z}') \\
&\quad + \delta^2 F/\delta\omega^J(\mathbf{x}')\, \delta\omega^P(\mathbf{z})\, \delta G/\delta\omega^L(\mathbf{y}')\, \delta F/\delta\omega^Q(\mathbf{z}')] \\
&\quad \times dx\, dx'\, dy\, dy'\, dz\, dz'
\end{aligned} \tag{6}$$

And from the right-hand side of (5) we have

$$\begin{aligned}
\mathscr{I}_2 &= \{\{F^2, G\}_+^*, F\}_+^* \\
&= 2 \int M^{IJ}(\mathbf{x}-\mathbf{x}')M^{KL}(\mathbf{y}-\mathbf{y}')M^{PQ}(\mathbf{z}-\mathbf{z}')\, \delta^2 F/\delta\omega^I(\mathbf{x})\, \delta\omega^L(\mathbf{y}') \\
&\quad \times [\delta F/\delta\omega^J(\mathbf{x}')\, \delta^2 G/\delta\omega^K(\mathbf{y})\, \delta\omega^Q(\mathbf{z}')\, \delta F/\delta\omega^P(\mathbf{z}') \\
&\quad + \delta^2 F/\delta\omega^J(\mathbf{x}')\, \delta\omega^P(\mathbf{z})\, \delta G/\delta\omega^L(\mathbf{y}')\, \delta\omega^Q(\mathbf{z}')] \\
&\quad \times dx\, dx'\, dy\, dy'\, dz\, dz'
\end{aligned} \tag{7}$$

Then, by a change of summation indices and of integration variables, we get from (6) and (7) that

$$\mathscr{I}_1 = \mathscr{I}_2$$

Hence we can conclude that if $\mathscr{L}$ is a degenerate functional Lagrangian such that Assumptions 1 and 2 are satisfied, the set of observable functionals of the corresponding field theory constitutes a Jordan algebra under the plus Dirac bracket combination law (3).

3. CONCLUDING REMARKS

We show that for a field theory satisfying two simple assumptions, the symmetric Dirac bracket defines a Jordan algebra on the associated set of observable functionals. It is interesting to note that the Assumptions 1 and 2 are not too restrictive, since the usual class of field theories for half-integer spin (Fermi) systems satisfies them. In this class of theories we have the Weyl, Dirac, and Rarita-Schwinger fields, for instance.

Nevertheless, fermionic string theories (Ramond, 1971; Neveu and Schwarz, 1971; Schwarz, 1982), do not satisfy Assumption 2 and, consequently, equation (5) is no longer automatically satisfied, but imposes some restrictions on the constraints. The question of whether the Jordan algebra is also present in such theories will be investigated in a future paper.

The result obtained in this note can be useful in the study of the quantization method for fields. In fact, knowing that a classical field defines a Jordan algebra, one has that the corresponding quantum theory must retain the algebraic structure of the dynamical variables (Hermann, 1970; T. F. Jordan and Sudarshan, 1961; Streater, 1966). Hence, the quantization procedure of such fields corresponds to obtaining representations of the Jordan algebra by operators in a Hilbert space. In that context it is expected that the study of representations of Jordan algebras in Hilbert spaces should be of physical significance.

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